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“Stabilization of Linear Systems with Structured Perturbations”

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Stabilization of Linear Systems with Structured Perturbations*

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Abstract

The problem of stabilization of linear systems with bounded structured uncertainties are considered in this paper. Two notions of stability, denoted quadratic stability (\mathcal{Q} -stability) and μ -stability, are considered, and corresponding notions of stabilizability and detectability are defined. In both cases, the output feedback stabilization problem is reduced via a separation argument to two simpler problems: full information (FI) and full control (FC). The set of all stabilizing controllers can be parametrized as a linear fractional transformation (LFT) on a free stable parameter. For \mathcal{Q} -stability, stabilizability and detectability can in turn be characterized by Linear Matrix Inequalities (LMIs), and the FI and FC \mathcal{Q} -stabilization problems can be solved using the corresponding LMIs. In the standard one-dimensional case the results in this paper reduce to well-known results on controller parametrization using state-space methods, although the development here relies more heavily on elegant LFT machinery and avoids the need for coprime factorizations.

1 Introduction

In this paper we are concerned with a class of linear systems which are represented as *linear fractional transformations* (LFTs) on some frequency/uncertainty structures:

$$G(\Delta) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] := D + C\Delta(I - A\Delta)^{-1}B$$

where the frequency/uncertainty structure Δ is defined as a set of complex matrices:

$$\Delta = \{\text{diag}[\delta_1 I_{r_1}, \dots, \delta_s I_{r_s}, \Delta_1, \dots, \Delta_f] : \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j \times m_j}\} \subset \mathbb{C}^{n \times n}.$$

There are many possible interpretations and applications for systems of this type, which will be referred to as LFT systems for simplicity. For a linear system with structured uncertainties, the first repeated scalar block $\delta_1 I_{r_1}$ of $\Delta \in \Delta$ can be viewed as a transform variable in

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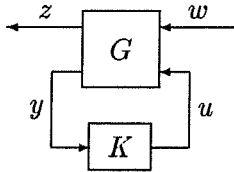
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a uncertain system with the remaining blocks of Δ viewed as norm-bounded perturbations. Alternatively, if $f = 0$, $\Delta \in \Delta$ may also be interpreted as transform variables in a multidimensional system. The various interpretations of Δ will only be considered briefly in this paper for motivation (see [21] for more detail on this setting).

This LFT notation is a direct generalization of the now standard notation for the state-space realizations of transfer functions. One of the advantages of the use of LFTs with this notation is that it facilitates manipulation using state-space-like machinery. Thus, we often refer to the “state” and “state transformations” of a system even when these terminologies do not, strictly speaking, apply. However, their meaning should be clear from context.

A basic feedback configuration considered in this paper is the following:



where G is the plant with two sets of inputs: the exogenous inputs w and the control inputs u , and with two sets of outputs: the measured outputs y and the regulated outputs z . The control problem is to design a feedback controller K such that the resulting closed loop system has some prescribed properties, in this case stability. In this setting, both G and K are LFTs on a frequency/uncertainty structure Δ and the controller is allowed to have the same dependence on the frequency/uncertainty structure as the plant. If Δ is viewed as uncertainty, then controller synthesis can be given a gain scheduling interpretation, as the controller depends on the same perturbations as does the plant. If the Δ is viewed as transform variables in a multidimensional system, the controller may be interpreted as employing dynamic feedback.

As part of the background material of this paper, we shall review some analysis results about LFT systems, particularly robust stability analysis, and then considers the associated synthesis problem of stabilization. (Some other related issues in this setting are considered in [15, 22, 20]) The stability notions employed in this paper are reasonably standard and are natural generalization of the conventional notions of stability [3], \mathcal{H}_∞ performance of discrete time systems [10, 21], and robust stability [10, 21]. Notions of stabilizability and detectability, which are related to the solvability of the stabilization problem, are also introduced. Two notions of robust stability are considered here, called μ and \mathcal{Q} stability. Each is a necessary and sufficient test for robust stability with respect to certain assumptions on the uncertainty Δ . Roughly speaking, μ treats LTI Δ and \mathcal{Q} treats LTV Δ . In this paper, we focus particularly on the \mathcal{Q} -case, where the synthesis problem is more tractable; the conditions for the \mathcal{Q} -stability, stabilizability and detectability can be characterized using LMIs, which result in convex optimization problems (See [10, 5] for surveys).

The approach to the stabilization and the stabilizing controller characterization problems is motivated by the techniques of Doyle et al [9] (see also [17]). The construction of stabilizing controllers for the *output feedback (OF)* problem is achieved via a separation argument which involves the reduction of the OF problem to two special problems: full information (FI)

problem and full control (FC) problem. The FI and FC \mathcal{Q} -stabilization problems are solved in terms of the positive definite solutions of certain LMIs, and the controllers can be chosen as static feedbacks. A resulting dynamic controller for the OF problem has a separation structure, and all stabilizing controllers are parametrized as a linear fractional transformation on the free stabilizing parameters.

The structure of this paper is as follows: In section 2, some background material and some examples for motivation are provided. In section 3, the properties of \mathcal{Q} -(μ -)stability, stabilizability and detectability are characterized. Two structural properties which will lead to a separation principle for LFT systems are described. In section 4, the main results about synthesis problems are stated; in addition, the static output feedback problem is considered. In section 5, the \mathcal{Q} -stabilization of the different special problems, FI , DF , FC and OE , are examined and the relationships among them are established. The output feedback problem is solved via separation arguments. In section 6, the stabilizing controller characterization problem is considered, and the parametrization of all stabilizing controllers is obtained from the special problems via separation arguments.

The following conventional notations will be adopted: \mathbb{R}^n and $\mathbb{R}^{n \times m}(\mathbb{C}^{n \times m})$ are the sets of n -dimensional real vectors and real(complex) matrices with dimension $n \times m$, respectively. \mathcal{RH}_∞ is the set of real rational functions analytic in the right half plane (or the unit disk). M^T denotes the transpose of M and M^* denotes the complex conjugate transpose of M . $\bar{\sigma}(M)$ denotes maximal singular value of the matrix M .

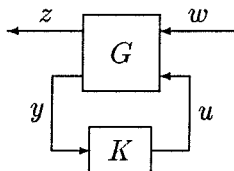
2 Preliminaries: LFTs and Linear Systems

In this section we review some standard material on analysis of systems described by LFTs. For additional background material on both linear fractional transformations(LFTs) and μ see [24, 25, 7, 11, 10], or the survey article [21].

2.1 LFTs and μ

Linear Fractional Transformations

The LFT formula arises naturally when we describe a well-posed feedback system as shown by the following block diagram.



The resulting input/output relation in the above diagram can be represented as $z = \mathcal{F}_l(G, K)w$, where $\mathcal{F}_l(G, K)$ is defined as the (lower) linear fractional transformation (LFT) on K with

the coefficient matrix G . Suppose G is partitioned conformally as

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \in \mathbb{C}^{(p_1+p_2) \times (q_1+q_2)}. \quad (1)$$

Then

$$\mathcal{F}_l(G, K) := G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \quad (2)$$

provided these inverses are well defined [24, 25]. If G_{21} is square and nonsingular, then $\mathcal{F}_l(G, K) = (A + BK)(C + DK)^{-1}$ with

$$A = G_{11}G_{21}^{-1}, \quad B = G_{12} - G_{11}G_{21}^{-1}G_{22}, \quad C = G_{21}^{-1}, \quad D = -G_{21}^{-1}G_{22}.$$

Similarly, the (upper) LFT on Δ , which corresponds to the feedback Δ around upper loop, is defined as

$$\mathcal{F}_u(G, \Delta) = G_{22} + G_{21}\Delta(I - G_{11}\Delta)^{-1}G_{12}. \quad (3)$$

The two LFT formulas are related as stated in the following proposition which can be verified directly from their definitions.

Proposition 1 *Given a LFT $\mathcal{F}_l(M, \Delta)$, there is a corresponding matrix N such that $\mathcal{F}_u(N, \Delta) = \mathcal{F}_l(M, \Delta)$ with $N = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} M \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, where the dimensions of the identity matrices are compatible with the partitions of M and N .*

Next, consider a well-posed closed loop system $P = \mathcal{F}_l(G, K)$ as shown in the above diagram, then

$$\begin{bmatrix} z \\ y \end{bmatrix} = G \begin{bmatrix} w \\ u \end{bmatrix}, \quad u = Ky$$

and

$$z = \mathcal{F}_l(G, K)w = Pw.$$

Now suppose G is an invertible transfer matrix. Then

$$\begin{bmatrix} w \\ u \end{bmatrix} = G^{-1} \begin{bmatrix} z \\ y \end{bmatrix}, \quad z = Pw$$

and

$$u = \mathcal{F}_u(G^{-1}, P)y$$

i.e. $K = \mathcal{F}_u(G^{-1}, P)$. This observation about the inversion property of a LFT can be summarized as the following proposition, whose proof can be found in [11].

Proposition 2 Suppose G is partitioned as in (1).

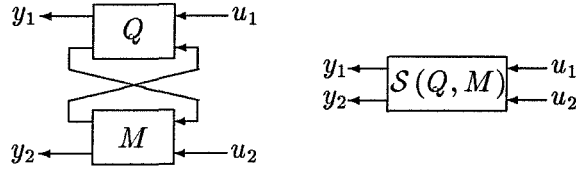
- (a) Assume G_{12} and G_{21} have full column and row rank, respectively, if matrices K_1 and K_2 are such that $\mathcal{F}_l(G, K_1) = \mathcal{F}_l(G, K_2)$ then $K_1 = K_2$.
- (b) Let $P = \mathcal{F}_l(G, K)$. If G , G_{12} and G_{21} are square and invertible, and $\det\left(G - \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}\right) \neq 0$, then $K = \mathcal{F}_u(G^{-1}, P)$.

Redheffer Star Products

Suppose that Q and M are complex matrices, suitably partitioned as

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \in \mathbb{C}^{(n_1+m_1) \times (n_2+m_2)}, \quad M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathbb{C}^{(m_2+l_1) \times (m_1+l_2)}.$$

with $n_1, n_2, l_1, l_2 \geq 0$. Consider the following two block diagrams,



If $I - Q_{22}M_{11}$ is invertible, then $\mathcal{S}(Q, M)$ is called the *Redheffer star product* of Q and M and is so defined that the above two block diagrams are equivalent, i.e.

$$\mathcal{S}(Q, M) := \begin{bmatrix} \mathcal{F}_l(Q, M_{11}) & Q_{12}(I - M_{11}Q_{22})^{-1}M_{12} \\ M_{21}(I - Q_{22}M_{11})^{-1}Q_{21} & \mathcal{F}_u(M, Q_{22}) \end{bmatrix}. \quad (4)$$

Note that for any compatibly dimensioned matrix K , we have

$$\mathcal{F}_l(\mathcal{S}(Q, M), K) = \mathcal{F}_l(Q, \mathcal{F}_l(M, K))$$

provided that the related LFTs are well-posed.

Structured Singular Values

Consider a matrix $M \in \mathbb{C}^{n \times n}$ and a underlying block structure Δ

$$\Delta := \{\text{diag}[\delta_1 I_{r_1}, \dots, \delta_s I_{r_s}, \Delta_1, \dots, \Delta_f] : \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j \times m_j}\} \subset \mathbb{C}^{n \times n}$$

where the subset $\mathbf{B}\Delta$ is defined as

$$\mathbf{B}\Delta := \{\Delta \in \Delta : \bar{\sigma}(\Delta) \leq 1\}.$$

Definition 1 The structured singular value $\mu_\Delta(M)$ of a matrix M with respect to a structure Δ is defined as

$$\mu_\Delta(M) := \sup_{\Delta \in \Delta} \left\{ \frac{1}{\bar{\sigma}(\Delta)} : \det[I - \Delta M] = 0 \right\} \quad (5)$$

unless no $\Delta \in \Delta$ makes $I - \Delta M$ singular, in which case $\mu_\Delta(M) := 0$.

Remark 1 From the definition, we have the following two special cases.

- 1) If $\Delta = \{\delta I : \delta \in \mathbb{C}\}$, then $\mu_\Delta(M) = \rho(M)$.
- 2) If $\Delta = \mathbb{C}^{n \times n}$, then $\mu_\Delta(M) = \bar{\sigma}(M)$.

For the structure Δ the commutative matrix set \mathcal{D} of Δ is defined as

$$\mathcal{D} = \{D \in \mathbb{C}^{n \times n} : D\Delta = \Delta D, \det[D] \neq 0, \Delta \in \Delta\}.$$

Thus \mathcal{D} depends only on the structure of Δ , and has the following properties.

Lemma 1 Let $D, D_1, D_2 \in \mathcal{D}$. Then $D^{-1} \in \mathcal{D}$, $D^* \in \mathcal{D}$ and $D_1 D_2 \in \mathcal{D}$.

Definition 2 The \mathcal{Q} -value of M with respect to a structure Δ is defined as

$$\mathcal{Q}_\Delta(M) := \inf_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1}). \quad (6)$$

Proposition 3 $\mathcal{Q}_\Delta(M)$ is an upper bound of $\mu_\Delta(M)$, i.e.

$$\mu_\Delta(M) \leq \mathcal{Q}_\Delta(M) = \inf_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1}).$$

Remark 2 Note that in general we have $\mu_\Delta(M) < \mathcal{Q}_\Delta(M)$. However, the equality holds for the following block structures:

- 1) $\Delta = \{\text{diag}[\delta I_r, \Delta] : \delta \in \mathbb{C}, \Delta \in \mathbb{C}^{(n-r) \times (n-r)}\}$
- 2) $\Delta = \{\text{diag}[\Delta_1, \dots, \Delta_f] : \Delta_i \in \mathbb{C}^{m_i \times m_i}\} \in \mathbb{C}^{n \times n}$, $f \leq 3$, where no blocks are repeated.

2.2 Examples of LFT Systems

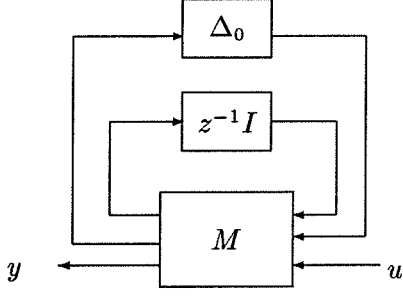
A large class of linear systems can be described in terms of *LFTs* on some specified frequency/uncertainty structures. The following examples serve to motivate the LFT descriptions.

Robust Stability of Systems with Structured Uncertainties

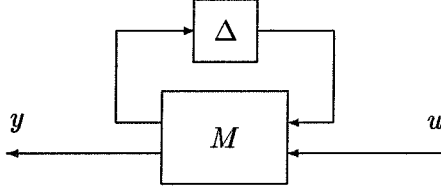
A uncertain discrete time linear system is considered in this subsection. Suppose that a nominal system described by the following equations

$$\begin{aligned} x(k+1) &= M_{11}x(k) + M_{13}u(k) \\ y(k) &= M_{31}x(k) + M_{33}u(k) \end{aligned}$$

is internally stable ($\rho(M_{11}) < 1$) and that a uncertainty set Δ_0 enters in a linear fractional way as shown in the following diagram, with $M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$.



Define $\Delta = \left\{ \begin{bmatrix} z^{-1}I & 0 \\ 0 & \Delta_0 \end{bmatrix} : z^{-1} \in \mathbb{C}, \Delta_0 \in \Delta_0 \right\}$. Then the uncertain system, which is a LFT on the frequency/uncertainty structure Δ , can be simply redrawn as the following diagram:



Let $A = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$. The test for robust stability depends on what further assumptions are made on the uncertainty $\Delta_0 \in \Delta_0$. If we assume that Δ_0 is either a constant complex matrix or a structured LTI operator, then we have robust stability if and only if $\mu_\Delta(A) < 1$. On the other hand, if Δ_0 is allowed to be an arbitrary time-varying operator, then we have robust stability if and only if $\mathcal{Q}_\Delta(A) < 1$. The sufficiency of the later condition is immediate from the small gain theorem and the necessity follows from the recent work by Shamma [27] and Megretski [18], see also citePaD for an overview.

Linear Shift-Invariant Multidimensional Systems

Consider a 2-dimensional discrete time linear shift invariant (LSI) system of order (n_1, n_2) which is described by the Roesser state space equations [26]:

$$\begin{aligned} x_1(k_1 + 1, k_2) &= A_{11}x_1(k_1, k_2) + A_{12}x_2(k_1, k_2) + B_1u(k_1, k_2) \\ x_2(k_1, k_2 + 1) &= A_{21}x_1(k_1, k_2) + A_{22}x_2(k_1, k_2) + B_2u(k_1, k_2) \\ y(k_1, k_2) &= C_1x_1(k_1, k_2) + C_2x_2(k_1, k_2) + Du(k_1, k_2) \end{aligned}$$

where $x_1(k_1, k_2) \in \mathbb{R}^{n_1}$ and $x_2(k_1, k_2) \in \mathbb{R}^{n_2}$ denote the system state vectors, $u(k_1, k_2) \in \mathbb{R}^p$ denotes the system input vector, and $y(k_1, k_2) \in \mathbb{R}^q$ denotes the system output vector. Note that the quadruple

$$(A, B, C, D) \in \mathbb{R}^{(n_1+n_2) \times (n_1+n_2)} \times \mathbb{R}^{(n_1+n_2) \times p} \times \mathbb{R}^{q \times (n_1+n_2)} \times \mathbb{R}^{q \times p}$$

with

$$A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C := \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

characterizes a LSI system of order (n_1, n_2) and with p inputs and q outputs.

The frequency structure Δ of this state space realization is defined as

$$\Delta := \left\{ \Delta = \begin{bmatrix} z_1^{-1}I & 0 \\ 0 & z_2^{-1}I \end{bmatrix} : \Delta \in \mathbb{C}^{(n_1+n_2) \times (n_1+n_2)} \right\}$$

where z_i denotes the forward shift operator. The transfer matrix for this system is

$$G(\Delta) = D + C \left(\begin{bmatrix} z_1I & 0 \\ 0 & z_2I \end{bmatrix} - A \right)^{-1} B = D + C\Delta(I - \Delta A)^{-1}B = \mathcal{F}_u \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta \right)$$

i.e. this system is a LFT with respect to frequency structure Δ .

More generally, consider a N -dimensional discrete LSI system with order (n_1, \dots, n_N) described as above in terms of a LFT with respect to frequency structure

$$\Delta = \{\text{diag}(z_1^{-1}I_{n_1}, \dots, z_N^{-1}I_{n_N}) : z_i \in \mathbb{C}\}.$$

Define

$$\bar{U}^N := \{(z_1, \dots, z_N) : |z_1| \geq 1, \dots, |z_N| \geq 1\}$$

and

$$T(z_1, \dots, z_N) = \det[I - \Delta A].$$

It is known that the N -dimensional system with system matrix A defined above is internally stable if $T(z_1, \dots, z_N) \neq 0$ in \bar{U}^N (cf. [3, 1]). Equivalently, the system is stable if and only if for any z_{10}, \dots, z_{N0} such that $T(z_{10}, \dots, z_{N0}) = 0$, then $\max\{|z_{10}^{-1}|, \dots, |z_{N0}^{-1}|\} < 1$. And the

system is stable if there exists $P = \text{diag}\{P_1, \dots, P_N\}$ positive definite, where $P_i \in \mathbb{R}^{n_i \times n_i}$, $i = 1, \dots, N$, such that the following Lyapunov inequality holds

$$APA^* - P < 0.$$

Note that the above internally stability definition is equivalent to $\mu_\Delta(A) < 1$, and the Lyapunov condition is equivalent to $\mathcal{Q}_\Delta(A) < 1$.

The stabilization problem in this setup is to design an output feedback controller $K(\Delta) = \mathcal{F}_u\left(\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}, \Delta\right)$ with the same frequency structure as plant, i.e. a dynamic feedback, such that the closed loop system is stable (cf. [3, 13]).

2.3 General LFT Systems

In the above discussions, we have examined a class of special systems which can be represented as LFTs on a block structure Δ in a subset Δ of $\mathbb{C}^{n \times n}$, i.e.

$$G(\Delta) = \mathcal{F}_u\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta\right)$$

with $(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times p} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times p}$. We will refer to this class of linear systems as *LFT systems*. For simplicity, we call the block structure Δ the frequency structure, recognizing that it has several alternative interpretations. For concreteness, assume that

$$\Delta = \{\text{diag}[\delta_1 I_{r_1}, \dots, \delta_s I_{r_s}, \Delta_1, \dots, \Delta_f] : \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{n_j \times m_j}\} \subset \mathbb{C}^{n \times n}.$$

All the results in the paper trivially extend to the more general case, including repeated full blocks, but the notation is cumbersome. The following notation is used to represent the LFT systems:

$$G(\Delta) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] := \mathcal{F}_u\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta\right).$$

By analogy with standard terminology, we will refer to this as “state space realization” of the transfer function $G(\Delta)$.

As in the conventional one-dimensional systems, (non-singular) state variable transformations are useful in the analysis and synthesis of LFT systems. But since not all transformations are allowed in this setting, the *admissible state variable transformations* has to be specified. Consider a LFT system

$$G(\Delta) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] := \mathcal{F}_u\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta\right)$$

with frequency structure Δ of dimension $n \times n$ and *commutative matrix set* \mathcal{D} of Δ .

If we think of the system as having “state” vector x , then a state variable transformation $x \mapsto x' := Tx$ is *admissible* if the transformation matrix $T \in \mathcal{D}$. The corresponding state space realization transformation is

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \mapsto \left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array} \right].$$

Note that the transfer function after the transformation does not change. In the next section we will examine some properties of LFT systems, and we will see that those properties are invariant under admissible state variable transformations.

3 Analysis of LFT Systems: The LMI Characterizations

In this section, we introduce some basic notions of LFT systems including stabilizability and detectability using both the μ and \mathcal{Q} notions of stability. The system under consideration is given by

$$G(\Delta) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

with frequency structure Δ , where $(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times p} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times p}$.

3.1 μ -Stability and \mathcal{Q} -Stability

Definition 3 *The linear system with matrix A and frequency structure Δ is μ -stable (with respect to Δ) if and only if $\mu_\Delta(A) < 1$.*

Remark 3 *Recalling the examples given in the last section, this definition is a generalization of the notions of stability for one or multi-dimensional discrete time systems, or stability with structured constant complex or LTI uncertainty.*

Definition 4 *A linear system with system matrix A and frequency structure Δ is quadratically stable (\mathcal{Q} -stable) (with respect to Δ) if and only if $\mathcal{Q}_\Delta(A) < 1$, i.e. there is a $D \in \mathcal{D}$ such that $\bar{\sigma}(DAD^{-1}) < 1$.*

Remark 4 *From this definition and Proposition 3 we can see that if a system is \mathcal{Q} -stable then it is μ -stable, but μ -stability doesn't imply \mathcal{Q} -stability in general. These two stability notions are equivalent if and only if $\mu_\Delta(A) = \mathcal{Q}_\Delta(A)$, for example in the case of one-dimensional systems with no uncertainty.*

The following theorem gives a characterization of the \mathcal{Q} -stability.

Theorem 1 *System A with frequency structure Δ is \mathcal{Q} -stable if and only if there exists a $P \in \mathcal{D}$ with $P = P^* > 0$ such that*

$$APA^* - P < 0 \tag{7}$$

where the matrix set \mathcal{D} is defined as in the last subsection.

Proof. System is \mathcal{Q} -stable iff there exists a $D \in \mathcal{D}$ such that $\bar{\sigma}(DAD^{-1}) < 1$ iff

$$DAD^{-1}(DAD^{-1})^* - I < 0$$

or

$$APA^* - P < 0$$

with $P = (D^*D)^{-1} \in \mathcal{D}, P = P^* > 0$. □

Remark 5 (i) The above condition is actually the Lyapunov condition with a structured positive definite matrix P . In particular, for a m -D system with order (n_1, \dots, n_m) , i.e. $A \in \mathbb{R}^{(n_1+\dots+n_m) \times (n_1+\dots+n_m)}$, if it is \mathcal{Q} -stable, then the Lyapunov condition in the above theorem is satisfied with $P = \text{diag}[P_1, \dots, P_m]$ and $0 < P_i \in \mathbb{R}^{n_i \times n_i}, i = 1, \dots, m$. The reader is referred to [1] for more equivalent conditions to the Lyapunov condition in the multidimensional case.

(ii) The above characterization gives an equivalent definition of \mathcal{Q} -stability of a given LFT system, this also motivates the notion of quadratic (\mathcal{Q} -)stability, since the above characterization is in quadratic form.

(iii) \mathcal{Q} -stability is a necessary and sufficient condition for robust stability with LTV perturbations [27, 18]

The following structural property of LFT systems follows immediately from the above definitions of μ -stability and \mathcal{Q} -stability and properties of μ :

Theorem 2 The μ -stability and \mathcal{Q} -stability of LFT systems are invariant under the admissible state variable transformations.

Another important structural property of LFT systems is expressed by the following theorem.

Theorem 3 Let A_1 and A_2 be two system matrices with respect to the frequency structures Δ_1 and Δ_2 , respectively. Then

- (i) if the system matrix $\begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix}$ with any compatibly dimensioned matrices A_{12} and A_{21} is \mathcal{Q} (or μ)-stable with respect to the frequency structure $\begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$, then A_1 and A_2 are also \mathcal{Q} (or μ)-stable with respect to structures Δ_1 and Δ_2 , respectively.
- (ii) the system matrix $\begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}$ with any compatibly dimensioned matrix A_{12} is \mathcal{Q} (or μ)-stable with respect to the frequency structure $\begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$ if and only if A_1 and A_2 are also \mathcal{Q} (μ)-stable with respect to structures Δ_1 and Δ_2 , respectively.

Proof. Assume that the commutative matrix sets of Δ_1 , Δ_2 and $\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$ are \mathcal{D}_1 , \mathcal{D}_2 and \mathcal{D} , respectively.

- (i) For the μ -case, these properties can be checked easily via the basic properties of μ . We will now focus on the \mathcal{Q} -case. Note that system $\begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix}$ is assumed to be \mathcal{Q} -stable, so there exists a positive definite $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \in \mathcal{D}$ (thus, $P_1 \in \mathcal{D}_1$ and $P_2 \in \mathcal{D}_2$ are both positive definite) such that

$$\begin{aligned} 0 < P - APA^* &= \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} - \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} A_1^* & A_{21}^* \\ A_{12}^* & A_2^* \end{bmatrix} \\ &= \begin{bmatrix} P_1 - A_1 P_1 A_1^* - A_{12} P_2 A_{12}^* & -A_1 P_1 A_{21}^* - A_{12} P_2 A_2^* \\ -A_{21} P_1 A_1^* - A_2 P_2 A_{12}^* & P_2 - A_2 P_2 A_2^* - A_{21} P_1 A_{21}^* \end{bmatrix}. \end{aligned}$$

This implies

$$P_1 - A_1 P_1 A_1^* \geq P_1 - A_1 P_1 A_1^* - A_{12} P_2 A_{12}^* > 0$$

and

$$P_2 - A_2 P_2 A_2^* \geq P_2 - A_2 P_2 A_2^* - A_{21} P_1 A_{21}^* > 0$$

which are what we need.

- (ii) The necessity was proved in part (i). The sufficiency is considered here. Note that the state transformation matrix $T = \begin{bmatrix} \alpha I & 0 \\ 0 & I \end{bmatrix}$ is admissible for some $\alpha \in \mathbb{R}$, and by conducting this transformation

$$A \mapsto TAT^{-1} = \begin{bmatrix} A_1 & dA_{12} \\ 0 & A_2 \end{bmatrix}$$

the transformed system matrix TAT^{-1} tends to $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ as d tends to 0. The latter system is stable as are A_1 and A_2 . By continuity of \mathcal{Q} or μ , the stability of resulting system is stable for some α close to 0. Therefore, A is stable since the admissible state transformation does not change stability. This argument holds for both μ and \mathcal{Q} stability.

□

Remark 6 Part (ii) of the above theorem also implies that a cascade system is \mathcal{Q} (or μ)-stable if and only if each subsystem is \mathcal{Q} (or μ)-stable.

3.2 Stabilizability and Detectability

Consider a LFT system $G(\Delta)$:

$$G(\Delta) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right].$$

The general stabilization problem is to design a (possibly dynamical) output feedback controller $K(\Delta_0)$ with a state-space realization

$$K(\Delta_0) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array} \right]$$

with frequency structure Δ_0 such that the feedback system is μ -stable or \mathcal{Q} -stable with respect to the induced new frequency structure $\Delta_N = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix}$. Here the structure Δ_0 is just some copies of Δ . The following lemma gives an equivalent description of the above stabilization problem.

Lemma 2 *The system $G(\Delta)$ can be μ -stabilized (or \mathcal{Q} -stabilized) by some $K(\Delta_0) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array} \right]$ with frequency structure Δ_0 related to Δ if and only if the augmented system*

$$G_a(\Delta_N) = \left[\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & 0 & 0 & I \\ \hline C & 0 & D & 0 \\ 0 & I & 0 & 0 \end{array} \right]$$

can be μ -stabilized (or \mathcal{Q} -stabilized) by static feedback $F = \begin{bmatrix} \hat{D} & \hat{C} \\ \hat{B} & \hat{A} \end{bmatrix}$ with respect to frequency structure $\Delta_N = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix}$.

Proof. This follows from the feedback-interconnection properties of LFTs. □

The stabilizability and detectability are defined in terms of the following two special structures, respectively,

$$G_{SF}(\Delta) = \left[\begin{array}{c|c} A & B \\ \hline I & 0 \end{array} \right] \quad G_{OI}(\Delta) = \left[\begin{array}{c|c} A & I \\ \hline C & 0 \end{array} \right]$$

where the frequency structures in both cases are the same as the one for $G(\Delta)$.

Definition 5 The system $G(\Delta)$ with frequency structure Δ is μ -stabilizable (or \mathcal{Q} -stabilizable) if there exists a dynamical controller for the corresponding system $G_{SF}(\Delta)$:

$$K(\Delta_0) = \mathcal{F}_u(F, \Delta_0) = \left[\begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array} \right]$$

such that the closed loop system is μ -stable (or \mathcal{Q} -stable) with respect to the induced frequency structure.

Definition 6 The system $G(\Delta)$ with frequency structure Δ is μ -detectable (or \mathcal{Q} -detectable) if there exists a dynamical controller for the corresponding system $G_{OI}(\Delta)$:

$$K(\Delta_0) = \mathcal{F}_u(L, \Delta_0) = \left[\begin{array}{c|c} L_{11} & L_{12} \\ \hline L_{21} & L_{22} \end{array} \right]$$

such that the closed loop system is μ -stable (or \mathcal{Q} -stable) with respect to the induced frequency structure.

We can characterize the two properties by the following two lemmas which follow from Lemma 2.

Lemma 3 The system $G(\Delta)$ is μ -stabilizable (or \mathcal{Q} -stabilizable), i.e., its corresponding system G_{SF} can be μ -stabilized (or \mathcal{Q} -stabilized) by some $K(\Delta_0) = \left[\begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array} \right]$ with frequency structure Δ_0 related to Δ if and only if the augmented system of $G_{SF}(\Delta)$

$$G_a(\Delta_N) = \left[\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & 0 & 0 & I \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array} \right]$$

can be μ -stabilized (or \mathcal{Q} -stabilized) by static feedback $F = \left[\begin{array}{cc} F_{22} & F_{21} \\ F_{12} & F_{11} \end{array} \right]$ with respect to frequency structure $\Delta_N = \left[\begin{array}{cc} \Delta & 0 \\ 0 & \Delta_0 \end{array} \right]$.

Lemma 4 The system $G(\Delta)$ is μ -detectable (or \mathcal{Q} -detectable), i.e. its corresponding system G_{OI} can be μ -stabilized (or \mathcal{Q} -stabilized) by some $K(\Delta_0) = \left[\begin{array}{c|c} L_{11} & L_{12} \\ \hline L_{21} & L_{22} \end{array} \right]$ with frequency structure Δ_0 related to Δ if and only if the augmented system of $G_{OI}(\Delta)$

$$G_a(\Delta_N) = \left[\begin{array}{cc|cc} A & 0 & I & 0 \\ 0 & 0 & 0 & I \\ \hline C & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array} \right]$$

can be μ -stabilized (or \mathcal{Q} -stabilized) by static injection $L = \begin{bmatrix} L_{22} & L_{21} \\ L_{12} & L_{11} \end{bmatrix}$ with respect to frequency structure $\Delta_N = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix}$.

3.3 LMI Characterizations of \mathcal{Q} -Stabilizability and \mathcal{Q} -Detectability

Let a system with frequency structure Δ be given by

$$G(\Delta) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

with $(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times p} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times p}$ and assume further that B and C are of full column and row rank, respectively, i.e., $\text{rank}(B) = p \leq n$ and $\text{rank}(C) = q \leq n$. Denote the commutative matrix set of Δ by \mathcal{D} .

For an one-dimensional system, the stabilizability (detectability) is equivalent to the fact that the system can be stabilized by a static state-feedback (output-injection). An immediate question is whether this property is still true for a general LFT system. We will have a positive answer for the \mathcal{Q} -case. But first, we shall consider how to characterize the static-state-feedback \mathcal{Q} -stabilizability.

In fact, if the above LFT system is \mathcal{Q} -stabilizable by a static state-feedback matrix $F \in \mathbb{R}^{p \times n}$, i.e. $\mathcal{Q}_\Delta(A + BF) < 1$, then by theorem 1, there exists a $P \in \mathcal{D}$ with $P = P^* > 0$ such that

$$(A + BF)P(A + BF)^* - P < 0.$$

If $\text{rank}(B) = p < n$ we can find a $B_\perp \in \mathbb{R}^{n \times (n-p)}$ such that $B^* B_\perp = 0$ and $\text{rank}(B_\perp) \leq n - p$, then we have

$$B_\perp^* (A + BF)P(A + BF)^* B_\perp - B_\perp^* P B_\perp < 0$$

or

$$B_\perp^* A P A^* B_\perp - B_\perp^* P B_\perp < 0.$$

So the solvability of the last LMI is necessary for the system to be static-state-feedback \mathcal{Q} -stabilizable. But, surprisingly, this condition is also sufficient if $\text{rank}(B_\perp) = n - p$ as stated by the following proposition.

Proposition 4 *Let $G(\Delta)$ be a LFT system with frequency structure Δ and $\text{rank}(B) = p < n$. Assume $B_\perp \in \mathbb{R}^{n \times (n-p)}$ is such that $B^* B_\perp = 0$ and $\begin{bmatrix} B & B_\perp \end{bmatrix}$ is invertible. Then there exists a static state feedback F such that $A + BF$ is \mathcal{Q} -stable with respect to the frequency structure Δ if and only if there exists a matrix $P \in \mathcal{D}$ with $P = P^* > 0$ such that*

$$B_\perp^* A P A^* B_\perp - B_\perp^* P B_\perp < 0. \tag{8}$$

Moreover, if P solves the above inequality, then the \mathcal{Q} -stabilizing static state feedback matrix can be chosen as

$$F = -(B^* P^{-1} B)^{-1} B^* P^{-1} A. \quad (9)$$

The proof of this proposition needs the following lemma:

Lemma 5 Assume $(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times p}$ and $\text{rank}(B) = p < n$. Let $B_\perp \in \mathbb{R}^{n \times (n-p)}$ and $B_0 \in \mathbb{R}^{p \times n}$ be such that $B_\perp^* B = 0$ and $\begin{bmatrix} B_0 & B_\perp \end{bmatrix}$ is unitary. Then

$$\inf_{F \in \mathbb{R}^{p \times n}} \bar{\sigma}(A + BF) = \bar{\sigma}(B_\perp^* A)$$

and the infimum is attained by $F = -(B_0^* B)^{-1} B_0^* A$.

Proof. Since $U := \begin{bmatrix} B_0 & B_\perp \end{bmatrix}$ is unitary,

$$\begin{aligned} \inf_{F \in \mathbb{R}^{p \times n}} \bar{\sigma}(A + BF) &= \inf_{F \in \mathbb{R}^{p \times n}} \bar{\sigma}(U^*(A + BF)) \\ &= \inf_{F \in \mathbb{R}^{p \times n}} \bar{\sigma} \left(\begin{bmatrix} B_0^* A + B_0^* B F \\ B_\perp^* A \end{bmatrix} \right) = \bar{\sigma} \left(\begin{bmatrix} 0 \\ B_\perp^* A \end{bmatrix} \right) = \bar{\sigma}(B_\perp^* A). \end{aligned}$$

Moreover the infimum is attained if $B_0^* A + B_0^* B F = 0$ or $F = -(B_0^* B)^{-1} B_0^* A$. \square

Remark 7 The matrix B_0 in Lemma 5 can be chosen as $B_0 = B(B^* B)^{-1/2}$, so in this case, $F = -(B^* B)^{-1} B^* A$.

Next, we prove Proposition 4.

Proof of Proposition 4. There exists a static feedback F such that the closed loop system matrix $A + BF$ is \mathcal{Q} -stable with respect to the frequency structure Δ if and only if

$$1 > \inf_{F, D} \bar{\sigma}(D(A + BF)D^{-1}) = \inf_{F, D} \bar{\sigma}(DAD^{-1} + DBFD^{-1})$$

where the infimum is over all possible $F \in \mathbb{R}^{p \times n}$ and $D \in \mathcal{D}$. Let $V_\perp^* = (B_\perp^* (D^* D)^{-1} B_\perp)^{-\frac{1}{2}} B_\perp^* D^{-1}$, then it is easy to check that $V_\perp^* V_\perp = I$ and $V_\perp^* (DB) = 0$. By Lemma 5, we have

$$1 > \inf_{F, D} \bar{\sigma}(D(A + BF)D^{-1}) = \inf_D \bar{\sigma}(V_\perp^* DAD^{-1})$$

or there exists a $D \in \mathcal{D}$ such that

$$(V_\perp^* DAD^{-1})(V_\perp^* DAD^{-1})^* < I.$$

Take $P = (D^* D)^{-1}$, then $P \in \mathcal{D}$ and $P = P^* > 0$, hence we have

$$(B_\perp^* P B_\perp)^{-\frac{1}{2}} B_\perp^* A P A^* B_\perp (B_\perp^* P B_\perp)^{-\frac{1}{2}} - I < 0$$

or

$$B_{\perp}^* A P A^* B_{\perp} - B_{\perp}^* P B_{\perp} < 0.$$

Moreover, if $P \in \mathcal{D}$ with $P = P^* > 0$ solves the above inequality, then we can construct a constant state feedback matrix F via Lemma 5 such that $A + BF$ is \mathcal{Q} -stable. Take $V_0^* = (B^*(D^*D)B)^{-1/2}B^*D^*$ with $[V_0, V_{\perp}]$ unitary, then $FD^{-1} = -(V_0^*DB)^{-1}V_0^*DAD^{-1}$, so

$$F = -(V_0^*DB)^{-1}V_0^*DA = -(B^*P^{-1}B)^{-1}B^*P^{-1}A.$$

□

Using the above result we can easily get

Theorem 4 *The system $G(\Delta)$ is \mathcal{Q} -stabilizable if and only if there exists a static feedback matrix F such that $A + BF$ is \mathcal{Q} -stable with respect to the same frequency structure.*

Proof. If B is square and of full rank, then the result is trivial. We only consider the case where $\text{rank}(B) = p < n$.

The sufficiency is obvious. As for the necessity, assume that the system can be \mathcal{Q} -stabilized by a dynamical controller $K(\Delta) = \mathcal{F}_u(F_0, \Delta_0)$ where $F_0 = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$ and Δ_0 is related to the system frequency structure Δ . By Lemma 3, this is equivalent to the fact that the augmented system

$$G_a(\Delta_N) = \left[\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & 0 & 0 & I \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array} \right]$$

is \mathcal{Q} -stabilized by the static feedback $\begin{bmatrix} F_{22} & F_{21} \\ F_{12} & F_{11} \end{bmatrix}$ with respect to the frequency structure

$\Delta_N = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix}$. Denote the commutative matrix set of Δ_N by \mathcal{D}_N , then by the above

proposition, there exists a $P_N =: \begin{bmatrix} P & P_1 \\ P_1^T & P_0 \end{bmatrix} \in \mathcal{D}_N$, which is positive definite such that

$$\begin{bmatrix} B_{\perp} \\ 0 \end{bmatrix}^* \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} P_N \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} B_{\perp} \\ 0 \end{bmatrix} - \begin{bmatrix} B_{\perp} \\ 0 \end{bmatrix}^* P_N \begin{bmatrix} B_{\perp} \\ 0 \end{bmatrix} < 0$$

i.e.

$$B_{\perp}^* A P A^* B_{\perp} - B_{\perp}^* P B_{\perp} < 0.$$

So the above LMI has a solution $P > 0$. It can be verified that $P \in \mathcal{D}$ by using the assumptions on the frequency structures and their commutative matrix sets. Therefore, the system can be \mathcal{Q} -stabilized by a static feedback matrix via the previous proposition. □

It can be seen that the \mathcal{Q} -stabilizability can be elegantly characterized in terms of a LMI; it is also the case for \mathcal{Q} -detectability by some dual arguments.

Proposition 5 *Let $G(\Delta)$ be a LFT system with frequency structure Δ and $\text{rank}(C) = q < n$. Assume that $C_\perp \in \mathbb{R}^{(n-q) \times n}$ is such that $C_\perp C^* = 0$ and $\begin{bmatrix} C \\ C_\perp \end{bmatrix}$ is invertible. Then there exists a static injection L such that $A + LC$ is \mathcal{Q} -stable with respect to the frequency structure Δ if and only if there exists a matrix $P \in \mathcal{D}$ with $P = P^* > 0$ such that*

$$C_\perp A^* P A C_\perp^* - C_\perp P C_\perp^* < 0. \quad (10)$$

Theorem 5 *The system $G(\Delta)$ is \mathcal{Q} -detectable if and only if there exists a static injection matrix L such that $A + LC$ is \mathcal{Q} -stable with respect to the same frequency structure. Moreover, if P is a solution to the LMI (10), then L can be taken as*

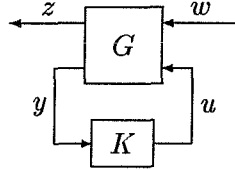
$$L = -AP^{-1}C^*(CP^{-1}C^*)^{-1}.$$

4 Synthesis of LFT Systems: Stabilization and Controller Characterization

In this section we state the main results for LFT system stabilization problems, their constructive proofs will be given in the next two sections.

4.1 Problem Statements and Assumptions

Consider the control system with standard block diagram



A general synthesis problem is to find a feedback mapping K such that the closed-loop system is well-posed and behaves well in some required sense.

Suppose $(w, u, z, y) \in \mathbb{R}^{p_1} \times \mathbb{R}^{p_2} \times \mathbb{R}^{q_1} \times \mathbb{R}^{q_2}$, and suppose $G(\Delta)$ with frequency/uncertainty structure Δ has a realization (with state $x \in \mathbb{R}^n$) as

$$G(\Delta) = \begin{bmatrix} G_{11}(\Delta) & G_{12}(\Delta) \\ G_{21}(\Delta) & G_{22}(\Delta) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right]$$

where all matrices are real and have compatible dimensions with the related physical variables. We further assume that $\text{rank}(B_2) = p_2 \leq n$ and $\text{rank}(C_2) = q_2 \leq n$. We will mainly consider

the non-trivial case where $p_2 < n$ and $q_2 < n$, but the solutions in other cases will be mentioned. In addition, let the state-space realization of $K(\Delta)$ be

$$K(\Delta_0) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array} \right]$$

with the frequency/uncertainty structure Δ_0 which is determined by Δ . In particular, the controller can have the same dependence on the frequency/uncertainty structure as the plant. In the uncertain system case the controller can be given a “gain scheduling” interpretation, as the controller depends on the same perturbations as does the plant; in the multidimensional system case, this means that the dynamical feedback controller is allowed. The *well-posedness* of this interconnection implies $I - D_{22}\hat{D}$ is invertible.

From now on, we will concentrate on the stabilization-related synthesis problems. We will mainly consider the case where there is no constraint on the controller’s frequency structure Δ_0 , i.e. Δ_0 can access all information about the plant’s frequency structure Δ . For the static controller case, which will be considered soon, no information about the plant frequency structure Δ is available to the controller; while such “stabilization” problem as \mathcal{H}_∞ -control of LFT systems, only partial information about plant frequency structure is available to its controllers, its solution is considered in [15] in some detail, see also [22].

In the rest of the paper, we will further focus on the \mathcal{Q} -stability. We will say that a feedback controller $K(\Delta)$ is *admissible* if it \mathcal{Q} -stabilizes $G(\Delta)$, (i.e. $\mathcal{F}_l(G(\Delta), K(\Delta_0))$ is \mathcal{Q} stable). For convenience, this general synthesis problem is called the *output feedback(OF)*.

Next, we define the admissible controller set as \mathcal{K} , i.e.

$$\mathcal{K} = \{K(\Delta) : \mathcal{F}_l(G(\Delta), K(\Delta_0)) \text{ is } \mathcal{Q} \text{ stable}\}.$$

And a subset \mathcal{K}_s of \mathcal{K} is defined as

$$\mathcal{K}_s = \{K \in \mathbb{R}^{p_2 \times q_2} : \mathcal{F}_l(G(\Delta), K) \text{ is } \mathcal{Q} \text{ stable}\}.$$

The following two synthesis problems are considered in this paper:

- Find a static or dynamical output feedback $K(\Delta) \in \mathcal{K}$ which \mathcal{Q} -stabilizes $G(\Delta)$.
- Characterize all controllers $K \in \mathcal{K}$ that \mathcal{Q} -stabilize G , or more specifically, find J such that $\mathcal{K} = \{\mathcal{F}_l(J, Q) : Q(\Delta) \text{ is } \mathcal{Q}\text{-stable}\}.$

Note that G is \mathcal{Q} -stabilizable by K if and only if G_{22} can be stabilized by K . Assume that (A, B_2) is \mathcal{Q} -stabilizable and (C_2, A) is \mathcal{Q} -detectable, which is both sufficient and necessary for the solvability of the stabilization problem for the output feedback structure where controllers are not required to be static. The realization of K will also be assumed throughout to be \mathcal{Q} -stabilizable and \mathcal{Q} -detectable if it is not a constant matrix.

4.2 Solutions to Synthesis Problems: Static Controllers

Under some strong conditions, the system can be \mathcal{Q} -stabilized by static output-feedback controllers; this subsection is devoted to this problem. The results in current forms are essentially from [8, 20]. The following lemma is key to our solutions (c.f. [23, 6, 8]).

Lemma 6 (i) (**Parrott's Theorem**) Assume $(X, B, C, A) \in \mathbb{R}^{n_1 \times m_1} \times \mathbb{R}^{n_1 \times m_2} \times \mathbb{R}^{n_2 \times m_1} \times \mathbb{R}^{n_2 \times m_2}$, then

$$\inf_{X \in \mathbb{R}^{n_1 \times m_1}} \bar{\sigma} \left(\begin{bmatrix} X & B \\ C & A \end{bmatrix} \right) = \max \left\{ \bar{\sigma} \left(\begin{bmatrix} C & A \end{bmatrix} \right), \bar{\sigma} \left(\begin{bmatrix} B \\ A \end{bmatrix} \right) \right\} =: \gamma_0$$

and the infimum can be achieved by $\hat{X} = -YA^*Z$, where Y and Z solve the matrix equations $Y(\gamma_0^2 I - A^*A)^{1/2} = B$ and $(\gamma_0^2 I - AA^*)^{1/2}Z = C$.

(ii) suppose $\gamma > \gamma_0$. The solutions X such that $\bar{\sigma} \left(\begin{bmatrix} X & B \\ C & A \end{bmatrix} \right) < \gamma$ are exactly those of the form

$$X = -YA^*Z + \gamma(I - YY^*)^{1/2}W(I - Z^*Z)^{1/2}$$

where Y and Z solve the matrix equations $Y(\gamma^2 I - A^*A)^{1/2} = B$ and $(\gamma^2 I - AA^*)^{1/2}Z = C$ and W is an arbitrary contraction ($\bar{\sigma}(W) < 1$).

This lemma implies

Lemma 7 Consider the triple $(A, B, C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times p} \times \mathbb{R}^{q \times n}$ with $\text{rank}(B) = p < n$ and $\text{rank}(C) = q < n$. Let $B_\perp \in \mathbb{R}^{n \times (n-p)}$ and $B_0 \in \mathbb{R}^{p \times n}$ be such that $B_\perp^* B = 0$ and $\begin{bmatrix} B_0 & B_\perp \end{bmatrix}$ is unitary, and let $C_\perp \in \mathbb{R}^{n-q \times n}$ and $C_0 \in \mathbb{R}^{n \times q}$ be such that $C_\perp C^* = 0$ and $\begin{bmatrix} C_0 \\ C_\perp \end{bmatrix}$ is unitary. Then

$$\inf_{F \in \mathbb{R}^{p \times q}} \bar{\sigma}(A + BFC) = \max\{\bar{\sigma}(B_\perp^* A), \bar{\sigma}(AC_\perp^*)\}.$$

Moreover, if $\max\{\bar{\sigma}(B_\perp^* A), \bar{\sigma}(AC_\perp^*)\} = \gamma_0$ then the above infimum can be attained by

$$F_0 = -(B_0^* B)^{-1}(X_0(\gamma_0) - B_0^* AC_0^*)(C_0 C^*)^{-1}$$

where

$$X_0(\gamma_0) = -B_0^* AC_\perp^* B_\perp AC_\perp^* (\gamma_0^2 I - B_\perp^* AC_\perp^* C_\perp AB_\perp)^{-1} B_\perp^* AC_0^*.$$

Furthermore, assume $\gamma > \gamma_0$ then the solutions to

$$\inf_{F \in \mathbb{R}^{p \times q}} \bar{\sigma}(A + BFC) < \gamma$$

are exactly of the form

$$F = -(B_0^* B)^{-1}(X(W) - B_0^* AC_0^*)(C_0 C^*)^{-1}$$

with $X(W)$ parameterized by

$$X(W) = X_0(\gamma) + \gamma(I - B(\gamma^2 I - A^*A)^{-1}B^*)^{1/2}W(I - C^*(\gamma^2 I - AA^*)^{-1}C)^{1/2}$$

where W is an arbitrary contraction matrix ($\bar{\sigma}(W) < 1$).

Proof. Note that

$$\bar{\sigma}(A + BFC) = \bar{\sigma} \left(\begin{bmatrix} B_0^* AC_0^* + B_0^* BFCC_0^* & B_0^* AC_\perp^* \\ B_\perp^* AC_0^* & B_\perp^* AC_\perp^* \end{bmatrix} \right),$$

then the results follow the preceding lemma. \square

Consider the system $G_{22} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array} \right]$. We give the following theorem which can be proved similarly to proposition 4 by using the above lemma (see also Packard et al, 1991).

Theorem 6 *Consider the given system with $\text{rank}(B_2) = p_2 < n$ and $\text{rank}(C_2) = q_2 < n$. Assume that $B_\perp \in \mathbb{R}^{n \times (n-p_2)}$ is such that $B_2^* B_\perp = 0$ and $\begin{bmatrix} B_2 & B_\perp \end{bmatrix}$ is invertible, and $C_\perp \in \mathbb{R}^{(n-q_2) \times n}$ is such that $C_\perp C_2^* = 0$ and $\begin{bmatrix} C_2 \\ C_\perp \end{bmatrix}$ is invertible. Then there exists an admissible static controller, i.e. $\mathcal{K}_s \neq \emptyset$, if and only if there exists a positive definite matrix $X \in \mathcal{D}$ such that the following two matrix inequalities hold:*

$$B_\perp^* A X A^* B_\perp - B_\perp^* X B_\perp < 0$$

$$C_\perp A^* X^{-1} A C_\perp^* - C_\perp X^{-1} C_\perp^* < 0.$$

Note that by the same procedure in the proof of Proposition 4, we can constructively get a \mathcal{Q} -stabilizing static controller and the static controller characterization in terms of the solutions of the above two matrix inequalities. Note also that in the trivial cases, i.e. when $p_2 = n$ or $q_2 = n$, this problem is reduced to the state-feedback or output-injection problem.

As stated in Lemma 2, every stabilizing problem with dynamic controllers can be transformed to the static controller case, so the solutions can be obtained by statically \mathcal{Q} -stabilizing its augmented system (see also Packard et al, 1991).

4.3 Solutions to Synthesis Problems: Dynamical Controllers

In this section we give the main results about the stabilization of LFT systems. The controllers needn't be static, the constructive proofs will be given in the next two sections.

Theorem 7 *Consider the given system G with $\text{rank}(B_2) = p_2 < n$ and $\text{rank}(C_2) = q_2 < n$. Assume that $B_\perp \in \mathbb{R}^{n \times (n-p_2)}$ is such that $B_2^* B_\perp = 0$ and $\begin{bmatrix} B_2 & B_\perp \end{bmatrix}$ is invertible, and that $C_\perp \in \mathbb{R}^{(n-q_2) \times n}$ is such that $C_\perp C_2^* = 0$ and $\begin{bmatrix} C_2 \\ C_\perp \end{bmatrix}$ is invertible. Then there exists an admissible controller, i.e. $\mathcal{K} \neq \emptyset$, if and only if there exist two positive definite matrices $X \in \mathcal{D}$ and $Y \in \mathcal{D}$ such that the following two LMIs hold:*

$$B_\perp^* A X A^* B_\perp - B_\perp^* X B_\perp < 0$$

and

$$C_{\perp} A^* Y A C_{\perp}^* - C_{\perp} Y C_{\perp}^* < 0.$$

Moreover, when the conditions hold, such a controller can be given by

$$K(\Delta) = \left[\begin{array}{c|c} \frac{A + B_2 F + L C_2 + L D_{22} F}{F} & -L \\ \hline & 0 \end{array} \right]$$

with the same frequency structure Δ as the plant where

$$F = -(B^* X^{-1} B)^{-1} B^* X^{-1} A \quad L = -A Y^{-1} C^* (C Y^{-1} C^*)^{-1}.$$

The controller given in this theorem has a separation structure, and is of the “observer form”, we will discuss its structure in the next section. The next theorem gives a characterization of \mathcal{K} .

Theorem 8 *Assume that the conditions in the last theorem are satisfied, then the admissible controller set can be characterized by:*

$$\mathcal{K} = \{\mathcal{F}_t(J(\Delta), Q(\Delta)) : Q(\Delta) \text{ is } \mathcal{Q}\text{-stable}\}$$

where

$$J = \left[\begin{array}{cc|cc} \frac{A + B_2 F + L C_2 + L D_{22} F}{F} & -L & B_2 + L D_{22} & \\ \hline & 0 & I & \\ -(C_2 + D_{22} F) & I & & -D_{22} \end{array} \right].$$

Remark 8 *If $p_2 = n$ or $q_2 = n$ then the corresponding LMI conditions in the theorems disappear since the existence and solutions of F or L can be obtained easily without solving the corresponding LMI. For example, if $p_2 = n$ then a corresponding constant state-feedback matrix can be $F = B_2^{-1}(A_F - A)$ where A_F is chosen such that $\mathcal{Q}_{\Delta}(A_F) < 1$, say $A_F = \alpha I$ for some $|\alpha| < 1$.*

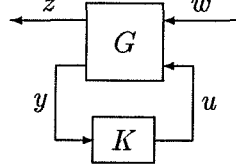
5 Stabilization Problem: Related Special Problems and A Construction

In this section we will consider the general stabilization problem which leads to a constructive proof of Theorem 7. Since the necessity is obvious, we only need to consider the sufficiency. The LMI conditions in Theorem 7 imply that there are constant matrices F and L such that $A + B_2 F$ and $A + L C_2$ are \mathcal{Q} -stable, and they are given in Theorem 4 and Theorem 5, so we now can assume this and do the constructions without being involved in solving any LMI at this stage. We first discuss four problems from which the solutions in Theorem 7 are constructed via a separation argument.

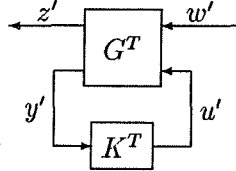
5.1 System Duality and Special Problems

It is well known that, in one-dimensional case, the concepts of controllability (stabilizability) and observability (detectability) of a system (C, A, B) are dual because of the duality between systems (C, A, B) and (B^T, A^T, C^T) . These dual notions can be generalized to the general feedback setting, and will play an important role in synthesis problems.

Consider a standard feedback system with block diagram



where the plant $G(\Delta)$ and the controller $K(\Delta_0)$ are assumed to be LFTs with respect to the frequency structure Δ and Δ_0 . Define another system shown below



whose plant and controller are obtained by transposing $G(\Delta)$ and $K(\Delta)$. It is routine to verify that $T_{zw}^T = [\mathcal{F}_\ell(G, K)]^T = \mathcal{F}_\ell(G^T, K^T) = T_{z'w'}$. It is also obvious that K \mathcal{Q} -stabilizes G with respect to the induced frequency structure $\Delta_N = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix}$ if and only if K^T \mathcal{Q} -stabilizes G^T with respect to the frequency structure Δ_N^T , or Δ_N , since Δ_N^T and Δ_N have the same structure. We say that these two control structures are *dual*, in particular, G^T and K^T are dual objects of G and K , respectively. As far as stabilization or other synthesis problems are concerned, we can obtain the results for G^T from its dual object G if available.

Next, consider some special problems which are related to the general OF problem. The special problems all pertain to the standard block diagram, but with different structures from G . The problems are labeled as

FI. Full information, with the corresponding plant

$$G_{FI}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \left[\begin{array}{c} I \\ 0 \end{array} \right] & \left[\begin{array}{c} 0 \\ I \end{array} \right] & \left[\begin{array}{c} 0 \\ 0 \end{array} \right] \end{array} \right]$$

FC. Full control, with the corresponding plant

$$G_{FC}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & \left[\begin{array}{cc} I & 0 \end{array} \right] \\ \hline C_1 & D_{11} & \left[\begin{array}{cc} 0 & I \end{array} \right] \\ C_2 & D_{21} & \left[\begin{array}{cc} 0 & 0 \end{array} \right] \end{array} \right]$$

DF. Disturbance feedforward, with the corresponding plant

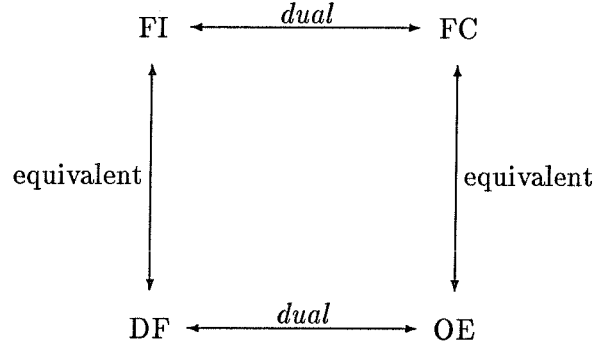
$$G_{DF}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & I & 0 \end{array} \right]$$

OE. Output estimation, with the corresponding plant

$$G_{OE}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & I \\ C_2 & D_{21} & 0 \end{array} \right]$$

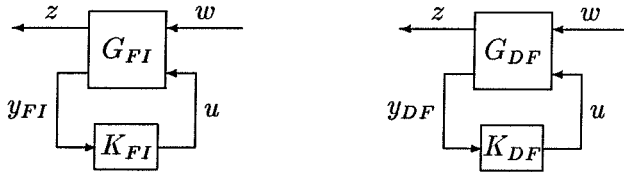
Note that all of these special systems have the same frequency structures as $G(\Delta)$. We assume all physical variables have the compatible dimensions. We say that they are special cases of the *OF* problem only in the sense that their structures are specified in comparing with the *OF* problems. The reader is referred to [9] for motivations of different problems.

It is clear that FI and FC (DF and OE) structures are dual, respectively; we will also see that FI and DF (FC and OE) structures are equivalent respectively in some sense that will be made precise in the next few subsections. These relationships are shown in the following diagram



5.2 FI and DF Problems

The connection between DF and FI problem is examined in this section. Suppose that we have controllers K_{FI} and K_{DF} connected to system as shown in the following diagrams,



Let T_{FI} and T_{DF} denote the closed-loop transfer matrices for the specified structures, respectively.

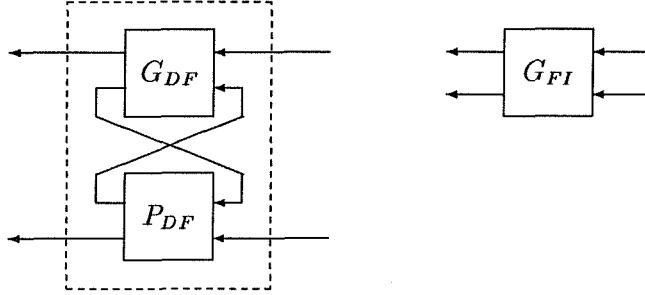
Proposition 6 Consider FI and DF structures as given in section 5.1. We have

$$(i) \ G_{DF}(\Delta) = \begin{bmatrix} I & 0 & 0 \\ 0 & C_2 & I \end{bmatrix} G_{FI}(\Delta)$$

(ii) $G_{FI} = \mathcal{S}(G_{DF}, P_{DF})$, where \mathcal{S} denotes the Redheffer star product and

$$P_{DF}(\Delta) = \left[\begin{array}{c|cc} A - B_1 C_2 & B_1 & B_2 \\ \hline 0 & 0 & I \\ \left[\begin{array}{c} I \\ -C_2 \end{array} \right] & \left[\begin{array}{c} 0 \\ I \end{array} \right] & \left[\begin{array}{c} 0 \\ 0 \end{array} \right] \end{array} \right].$$

Proof. (i) is easy, we only prove (ii). We need to prove that the two transfer functions shown in the following diagrams are the same, from which this theorem follows immediately.



Next, the first system is examined, let x and \hat{x} denote the state of G_{DF} and P_{DF} , respectively, take $e := x - \hat{x}$ and \hat{x} as the states of the resulting interconnected system, then its realization is

$$\left[\begin{array}{cc|cc} A - B_1 C_2 & 0 & 0 & 0 \\ B_1 C_2 & A & B_1 & B_2 \\ \hline C_1 & C_1 & D_{11} & D_{12} \\ \left[\begin{array}{c} 0 \\ C_2 \end{array} \right] & \left[\begin{array}{c} I \\ 0 \end{array} \right] & \left[\begin{array}{c} 0 \\ I \end{array} \right] & \left[\begin{array}{c} 0 \\ 0 \end{array} \right] \end{array} \right]$$

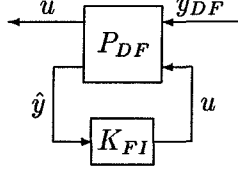
with respect to the frequency structure $\begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}$. The resulting transfer matrix is exactly G_{FI} , as claimed. \square

The following theorem follows immediately:

Theorem 9 (i) $K_{FI} := K_{DF} \begin{bmatrix} C_2 & I \end{bmatrix}$ \mathcal{Q} -stabilizes G_{FI} if K_{DF} \mathcal{Q} -stabilizes G_{DF} . Furthermore,

$$\mathcal{F}_\ell(G_{FI}, K_{DF} \begin{bmatrix} C_2 & I \end{bmatrix}) = \mathcal{F}_\ell(G_{DF}, K_{DF}).$$

- (ii) Suppose that $A - B_1C_2$ is \mathcal{Q} -stable. Then $K_{DF} := \mathcal{F}_\ell(P_{DF}, K_{FI})$ \mathcal{Q} -stabilizes G_{DF} if K_{FI} \mathcal{Q} -stabilizes G_{FI} .



Furthermore, $\mathcal{F}_\ell(G_{DF}, \mathcal{F}_\ell(P_{DF}, K_{FI})) = \mathcal{F}_\ell(G_{FI}, K_{FI})$.

Proof. (i) it is easy. As for (ii), note that by Proposition 6, we have

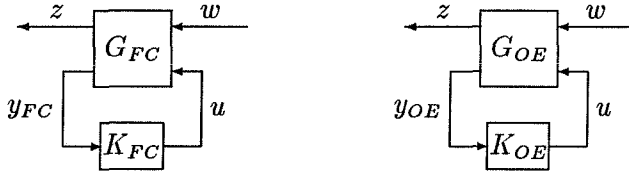
$$\mathcal{F}_\ell(G_{FI}, K_{FI}) = \mathcal{F}_\ell(\mathcal{S}(G_{FI}, P_{FD}), K_{FI}) = \mathcal{F}_\ell(G_{DF}, \mathcal{F}_\ell(P_{DF}, K_{FI})),$$

the \mathcal{Q} -stability of the latter is guaranteed by the stability of $A - B_1C_2$ and the choice of K_{FI} . \square

Remark 9 This theorem shows that if $A - B_1C_2$ is \mathcal{Q} -stable, then problems FI and DF are input/output equivalent. Since the stabilizing controllers for either structure can be obtained from the other such that the resulting input/output properties are the same.

5.3 FC and OE Problems

Consider the following FC and OE feedback structures



Proposition 7 Let FC and OE structures be given as in section 5.1. We have

$$(i) \ G_{OE}(\Delta) = G_{FI}(\Delta) \begin{bmatrix} I & 0 \\ 0 & B_2 \\ 0 & I \end{bmatrix}$$

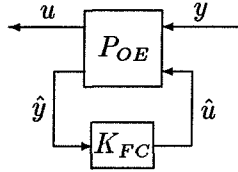
(ii) $G_{FC} = \mathcal{S}(G_{OE}, P_{OE})$, where P_{OE} is

$$P_{OE}(\Delta) = \left[\begin{array}{c|c|c} A - B_2C_1 & 0 & \begin{bmatrix} I & -B_2 \end{bmatrix} \\ \hline C_1 & 0 & \begin{bmatrix} 0 & I \end{bmatrix} \\ C_2 & I & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array} \right].$$

Theorem 10 (i) $K_{FC} := \begin{bmatrix} B_2 \\ I \end{bmatrix} K_{OE}$ \mathcal{Q} -stabilizes G_{FC} if K_{OE} \mathcal{Q} -stabilizes G_{OE} . Furthermore,

$$\mathcal{F}_l(G_{FC}, \begin{bmatrix} B_2 \\ I \end{bmatrix} K_{OE}) = \mathcal{F}_l(G_{OE}, K_{OE}).$$

(ii) Suppose that $A - B_2 C_1$ is \mathcal{Q} -stable. Then $K_{OE} := \mathcal{F}_l(P_{OE}, K_{FC})$ \mathcal{Q} -stabilizes G_{OE} if K_{FC} \mathcal{Q} -stabilizes G_{FC} .



Furthermore, $\mathcal{F}_l(G_{OE}, \mathcal{F}_l(P_{OE}, K_{FC})) = \mathcal{F}_l(G_{FC}, K_{FC})$.

Remark 10 This theorem shows that if $A - B_2 C_1$ is \mathcal{Q} -stable, then FC and OE problems are input/output equivalent.

5.4 OF Problem and Separation Property

In this section we constructively prove Theorem 7. Since the necessity is clear, we only consider the sufficiency. The construction essentially involves reducing the OF problem to a combination of the simpler FI and FC problems with the separation argument as the byproduct.

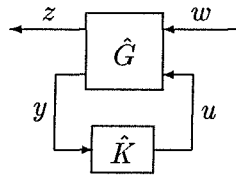
Without loss of generality, we shall assume $D_{22} = 0$. Since for more general case, i.e., $D_{22} \neq 0$ in the realization of $G(\Delta)$:

$$G(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right],$$

the mapping

$$\hat{K}(\Delta) = K(\Delta)(I - D_{22}K(\Delta))^{-1} = \mathcal{F}_l\left(\begin{bmatrix} 0 & I \\ I & D_{22} \end{bmatrix}, K(\Delta)\right)$$

is well defined by the assumption that the closed-loop system is well-posed. Therefore, the system in terms of \hat{K} has the structure



with

$$\hat{G}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right]$$

If \hat{K} is designed from the above structure, then K can be obtained from Proposition 2 as

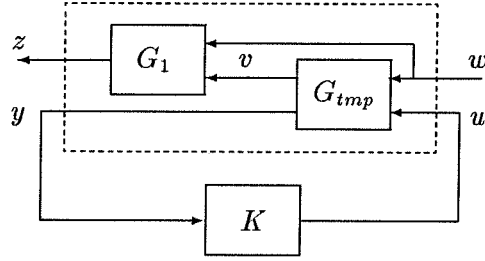
$$K(\Delta) = \mathcal{F}_u \left(\begin{bmatrix} -D_{22} & I \\ I & 0 \end{bmatrix}, \hat{K}(\Delta) \right) = \mathcal{F}_l \left(\begin{bmatrix} 0 & I \\ I & -D_{22} \end{bmatrix}, \hat{K}(\Delta) \right)$$

This justifies the simplification.

Now we construct the controllers for OF problem with $D_{22} = 0$. Let x denote the state of the system $G(\Delta)$. Since (A, B_2) is \mathcal{Q} -stabilizable, there is a constant matrix F such that $A + B_2F$ is \mathcal{Q} -stable. Note that $\begin{bmatrix} F & 0 \end{bmatrix}$ is actually a special FI stabilizing controller. Let

$$v = u - Fx$$

Then the system can be broken into two subsystems G_1 and G_{tmp} as shown pictorially below



with

$$G_1(\Delta) = \left[\begin{array}{c|cc} A + B_2F & B_1 & B_2 \\ \hline C_1 + D_{12}F & D_{11} & D_{12} \end{array} \right]$$

which is \mathcal{Q} -stable, and

$$G_{tmp}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline -F & 0 & I \\ C_2 & D_{21} & 0 \end{array} \right].$$

Since G_1 is \mathcal{Q} -stable, by Theorem 3 K \mathcal{Q} -stabilizes G if and only if K \mathcal{Q} -stabilizes G_{tmp} .

Note that G_{tmp} is of OE structure. Let L be such that $A + LC_2$ is \mathcal{Q} -stable then $\begin{bmatrix} L \\ 0 \end{bmatrix}$ is a \mathcal{Q} -stabilizing controller for the corresponding FC problem, since $A + B_2F$ is \mathcal{Q} -stable by construction, by Theorem 10 (ii) we have a controller given by

$$K(\Delta) = \mathcal{F}_l(J, \begin{bmatrix} L \\ 0 \end{bmatrix})$$

where

$$J(\Delta) = \left[\begin{array}{c|c} A + B_2F & 0 \\ \hline -F & 0 \\ C_2 & I \end{array} \left| \begin{array}{c} I \quad -B_2 \\ 0 \quad I \\ 0 \quad 0 \end{array} \right. \right].$$

Then we have

$$K(\Delta) = \left[\begin{array}{c|c} A + B_2F + LC_2 & -L \\ \hline F & 0 \end{array} \right].$$

Now we drop the assumption $D_{22} = 0$ and get the following result which restates Theorem 7.

Proposition 8 *Consider the general OF problem. Let F and L be such that $A + LC_2$ and $A + B_2F$ are \mathcal{Q} -stable, then the controller*

$$K(\Delta) = \left[\begin{array}{c|c} A + B_2F + LC_2 + LD_{22}F & -L \\ \hline F & 0 \end{array} \right]$$

with the frequency structure Δ \mathcal{Q} -stabilizes the given system.

The above construction was conducted by reducing the synthesis of OF problem to the independent synthesis of FI and OE problems. This reduction is based on the separation property. And it also leads to a separation structure for the resulting closed loop system.

We now take the state variable of the closed loop system as $\bar{x} = \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$, and the corresponding realization is

$$\left[\begin{array}{cc|c} A & B_2F & B_1 \\ -LC_2 & A + B_2F + LC_2 & -LD_{21} \\ \hline C_1 & D_{12}F & D_{11} \end{array} \right].$$

Next, we conduct the admissible state transformation $\bar{x} \mapsto T\bar{x} = \begin{bmatrix} x \\ \tilde{x} - x \end{bmatrix}$, i.e., $T =$

$\begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}$. After the transformation, the realization is

$$\left[\begin{array}{cc|c} A + B_2F & B_2F & B_1 \\ 0 & A + LC_2 & B_1 - LD_{21} \\ \hline C_1 - D_{12}F & D_{12}F & D_{11} \end{array} \right]$$

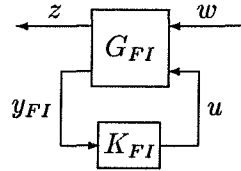
i.e. the system is decoupled into two separated \mathcal{Q} -stable subsystems, i.e. state-feedback system and output-injection system. Hence the closed-loop system after the admissible state variable transformation is also \mathcal{Q} -stable with respect to the new frequency structure $\Delta_N = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}$ by Theorem 3, so is the original closed-loop system as desired.

6 Stabilizing Controller Parameterization: A Construction

This section is mainly devoted to the proof of Theorem 8, i.e. to construct the parametrization of all admissible controllers. We follow [9, 17] to present a state-space-like approach to this problem without using any idea from coprime factorization. The techniques to be used are from the LFT theory, especially the inversion property of a LFT is often used. The main idea of this approach is similar to the one for stabilization problem. That is, we will reduce the OF problem into the simpler FI and OE problems, then solve the output feedback problem by separation argument. The emphasis of this section is on building up enough tools for this objective. For this purpose, we just parameterize equivalent controller classes for each special problem. Two controllers, K and K' , are said to be equivalent if they produce the same input/output relationships for the corresponding closed loop systems, i.e. $\mathcal{F}_l(G, K) = \mathcal{F}_l(G, K')$, written as $K \cong K'$.

6.1 Admissible Controllers for FI and FC Problems

Examine the *FI* structure:



where the transfer matrix G_{FI} is given in the last section. Note that the controllers for FI structure have the following general form

$$K_{FI}(\Delta) = \begin{bmatrix} K_1(\Delta) & K_2(\Delta) \end{bmatrix}$$

with $K_1(\Delta)$ \mathcal{Q} -stabilizing $\left[\begin{array}{c|c} A & B_2 \\ \hline I & 0 \end{array} \right]$ and arbitrary \mathcal{Q} -stable $K_2(\Delta)$.

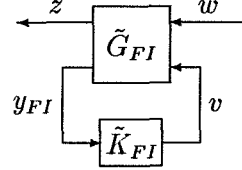
Proposition 9 *Let F be a constant matrix such that $A+B_2F$ is \mathcal{Q} -stable. Then all admissible controllers, in the sense of generating all \mathcal{Q} -stabilizing control, for FI can be parameterized as*

$$K_{FI}(\Delta) \cong \begin{bmatrix} F & Q(\Delta) \end{bmatrix}$$

with any \mathcal{Q} -stable $Q(\Delta)$.

Proof. It is easy to see that the controller given in the above formula \mathcal{Q} -stabilizes the system $G_{FI}(\Delta)$. Hence we only need to show that the given set of controllers parameterizes all equivalence classes of \mathcal{Q} -stabilizing controllers. So it is enough to show that there is a choice of \mathcal{Q} -stable $Q(\Delta)$ such that the transfer functions from w to u for any stabilizing controller $K_{FI}(\Delta) = \begin{bmatrix} K_1(\Delta) & K_2(\Delta) \end{bmatrix}$ and for $K_{FI}^0(\Delta) = \begin{bmatrix} F & Q(\Delta) \end{bmatrix}$ are the same, since

this implies $\mathcal{F}_l(G_{FI}, K_{FI}) = \mathcal{F}_l(G_{FI}, K_{FI}^0)$. To show that, make a change of control variable $v = u - Fx$, where x denotes the state of the system $G_{FI}(\Delta)$, then the system with the controller $K_{FI}(\Delta)$ is shown as in the following diagram:

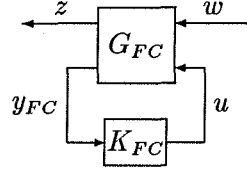


where

$$\tilde{G}_{FI} := \left[\begin{array}{c|cc} A + B_2 F & B_1 & B_2 \\ \hline C_1 + D_{12} F & 0 & D_{12} \\ \hline I & 0 & 0 \\ 0 & I & 0 \end{array} \right], \quad \tilde{K}_{FI} := K_{FI} - [F \ 0].$$

Let $Q(\Delta)$ be the transfer matrix from w to v ; it is \mathcal{Q} -stable by the \mathcal{Q} -stability of the closed loop system. Then $u = Fx + v = Fx + Qw$, so $K_{FI}(\Delta) \cong \begin{bmatrix} F & Q(\Delta) \end{bmatrix}$. \square

Next, the FC problem is considered



where G_{FC} is given in the last section. Dually, we have

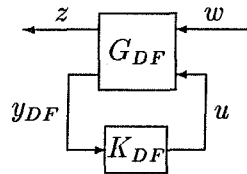
Proposition 10 *Let L be a constant matrix such that $A + LC_2$ is \mathcal{Q} -stable. Then the set of equivalent classes of all admissible controllers for FC in the above sense can be parameterized as*

$$K_{FC}(\Delta) \cong \begin{bmatrix} L \\ Q(\Delta) \end{bmatrix}$$

with any \mathcal{Q} -stable $Q(\Delta)$.

6.2 Admissible Controllers for Problems DF and OE

Consider the DF structure



The transfer matrix is given as in the last section. We will further assume that $A - B_1C_2$ is \mathcal{Q} -stable in this subsection. It should be pointed out that this assumption is not necessary for DF problem to be solvable, however it does simplify the solution. And of course it can be easily relaxed.

Note that under the above assumption, FI and DF problems are equivalent as pointed out in the last section, it can be show that if $K_{DF} \cong K'_{DF}$ in the DF structure, then $K_{DF} \begin{bmatrix} C_2 & I \end{bmatrix} \cong K'_{DF} \begin{bmatrix} C_2 & I \end{bmatrix}$ in the corresponding FI structure. Also if $K_{FI} \cong K'_{FI}$, then $\mathcal{F}_l(P_{DF}, K_{FI}) \cong \mathcal{F}_l(P_{DF}, K'_{FI})$.

Next, the parametrization of DF controllers is considered. Let $K_{DF}(\Delta)$ be an admissible controller for DF then $K_{FI}(\Delta) = K_{DF}(\Delta) \begin{bmatrix} C_2 & I \end{bmatrix}$ \mathcal{Q} -stabilizes the corresponding $G_{FI}(\Delta)$. Assume $K_{FI}(\Delta) \cong K'_{FI}(\Delta) = \begin{bmatrix} F & Q(\Delta) \end{bmatrix}$ for some \mathcal{Q} -stable $Q(\Delta)$, then $K'_{FI}(\Delta)$ \mathcal{Q} -stabilizes $G_{FI}(\Delta)$ and $\mathcal{F}_l(J_{DF}(\Delta), Q(\Delta)) = \mathcal{F}_l(P_{DF}(\Delta), K'_{FI}(\Delta))$ where

$$J_{DF}(\Delta) = \left[\begin{array}{c|cc} A + B_2F - B_1C_2 & B_1 & B_2 \\ \hline F & 0 & I \\ -C_2 & I & 0 \end{array} \right]$$

with F such that $A + B_2F$ is \mathcal{Q} -stable. Hence by Theorem 9, $K'_{DF}(\Delta) := \mathcal{F}_l(J_{DF}(\Delta), Q(\Delta))$ stabilizes $G_{DF}(\Delta)$ for any \mathcal{Q} -stable $Q(\Delta)$. Since $K_{FI}(\Delta) \cong K'_{FI}(\Delta)$, we have $K_{DF}(\Delta) \cong K'_{DF}(\Delta) = \mathcal{F}_l(J_{DF}(\Delta), Q(\Delta))$, which characterizes the equivalence classes of all controllers for DF problem by the equivalence of FI and DF .

Actually, as stated in the following proposition, the above construction of parametrization characterizes all admissible controllers (not just the equivalence classes) for DF problem.

Proposition 11 *All admissible controllers for the DF problem can be characterized by $K_{DF}(\Delta) = \mathcal{F}_l(J_{DF}(\Delta), Q_0(\Delta))$ with \mathcal{Q} -stable $Q_0(\Delta)$, where $J_{DF}(\Delta)$ is given as above.*

Proof. It is easy to show that the controllers expressed in the given LFT formula do \mathcal{Q} -stabilize G_{DF} by transforming it to the corresponding FI problem. Let K_{DF} be any admissible controller for G_{DF} , then $\mathcal{F}_l(\hat{J}_{DF}, K_{DF})$ is \mathcal{Q} -stable where

$$\hat{J}_{DF} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline -F & 0 & I \\ C_2 & I & 0 \end{array} \right].$$

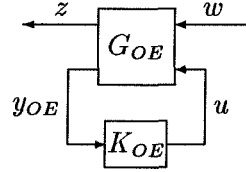
Let $Q_0 := \mathcal{F}_l(\hat{J}_{DF}, K_{DF})$, then $\mathcal{F}_l(J_{DF}, Q_0) = \mathcal{F}_l(J_{DF}, \mathcal{F}_l(\hat{J}_{DF}, K_{DF})) =: \mathcal{F}_l(J_{tmp}, K_{DF})$, where J_{tmp} can be obtained as

$$J_{tmp} = \left[\begin{array}{cc|cc} A - B_1C_2 + B_2F & -B_2F & B_1 & B_2 \\ -B_1C_2 & A & B_1 & B_2 \\ \hline F & -F & 0 & I \\ -C_2 & C_2 & I & 0 \end{array} \right]$$

$$\begin{aligned}
&= \left[\begin{array}{cc|cc} A - B_1 C_2 & -B_2 F & B_1 & B_2 \\ 0 & A + B_2 F & 0 & 0 \\ \hline 0 & -F & 0 & I \\ 0 & C_2 & I & 0 \end{array} \right] \\
&= \left[\begin{array}{cc} 0 & I \\ I & 0 \end{array} \right].
\end{aligned}$$

Hence $\mathcal{F}_\ell(J_{DF}, Q_0) = \mathcal{F}_\ell(J_{tmp}, K_{DF}) = K_{DF}$. This shows that any admissible controller can be expressed in the form of $\mathcal{F}_\ell(J_{DF}, Q_0)$ for some Q -stable Q_0 . \square

Next, we turn to the OE problem



G_{OE} is given in the last section. Similarly, we will assume that $A - B_2 C_1$ is Q -stable.

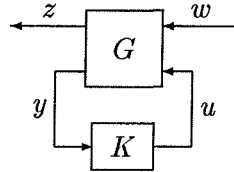
Proposition 12 *All admissible controllers for the OE problem can be characterized as $\mathcal{F}_\ell(J_{OE}, Q_0)$ with any Q -stable Q_0 , where J_{OE} is defined as*

$$J_{OE} = \left[\begin{array}{cc|cc} A - B_2 C_1 + L C_2 & L & -B_2 \\ \hline C_1 & 0 & I \\ C_2 & I & 0 \end{array} \right]$$

with L such that $A + L C_2$ is Q -stable.

6.3 All Admissible Controllers for Problem OF

The following standard system block diagram is considered again



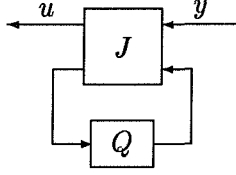
with

$$G(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right].$$

As before, it is assumed that (A, B_2) is Q -stabilizable and (C_2, A) is Q -detectable with respect to the frequency structure Δ .

We are now going to prove Theorem 8 which is restated as

Proposition 13 *Let F and L be such that $A + LC_2$ and $A + B_2F$ are \mathcal{Q} -stable, then all controllers which \mathcal{Q} -stabilize $G(\Delta)$ can be parameterized as the transfer function from y to u below*



$$J(\Delta) = \left[\begin{array}{c|cc} A + B_2F + LC_2 + LD_{22}F & -L & B_2 + LD_{22} \\ \hline F & 0 & I \\ \hline -(C_2 + D_{22}F) & I & -D_{22} \end{array} \right]$$

with any \mathcal{Q} -stable $Q(\Delta)$ such that the resulting closed loop system is well-posed.

Proof. We will assume again $D_{22} = 0$ for simplicity. Let x denote the state of system G . Since (A, B_2) is \mathcal{Q} -stabilizable, there is a constant matrix F such that $A + B_2F$ is \mathcal{Q} -stable. Note that $\begin{bmatrix} F & 0 \end{bmatrix}$ is actually a special FI \mathcal{Q} -stabilizing controller. Let

$$v = u - Fx$$

as in the proof of Proposition 8, thus $K(\Delta)$ \mathcal{Q} -stabilizes $G(\Delta)$ if and only if it \mathcal{Q} -stabilizes

$$G_{tmp}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline -F & 0 & I \\ \hline C_2 & D_{21} & 0 \end{array} \right].$$

However, $G_{tmp}(\Delta)$ is of the OE structure. Let L be such that $A + LC_2$ is \mathcal{Q} -stable. Then by Theorem 10 all controllers \mathcal{Q} -stabilizing $G_{tmp}(\Delta)$ are given by

$$K(\Delta) = \mathcal{F}_\ell(J(\Delta), Q(\Delta))$$

where

$$J(\Delta) = \left[\begin{array}{c|cc} A + B_2F + LC_2 & L & -B_2 \\ \hline -F & 0 & I \\ \hline C_2 & I & 0 \end{array} \right] = \left[\begin{array}{c|cc} A + B_2F + LC_2 & -L & B_2 \\ \hline F & 0 & I \\ \hline -C_2 & I & 0 \end{array} \right].$$

This concludes our proof. □

This theorem shows that any admissible controller $K(\Delta)$ can be characterized as a *LFT* of a \mathcal{Q} -stable parameter matrix $Q(\Delta)$, i.e., $K(\Delta) = \mathcal{F}_\ell(J(\Delta), Q(\Delta))$. There is an alternative direct proof that this parametrization produces *all* stabilizing controllers. To see this, recall from the inversion formulas for LFTs that we can solve the equation $K(\Delta) = \mathcal{F}_\ell(J(\Delta), Q(\Delta))$ for $Q(\Delta)$ to give

$$Q = \mathcal{F}_u(J^{-1}, K) = \mathcal{F}_\ell(\hat{J}, K)$$

where a little algebra shows that

$$J^{-1} = \left[\begin{array}{c|cc} A & B_2 & L \\ \hline C_2 & D_{22} & I \\ -F & I & 0 \end{array} \right]$$

and

$$\hat{J} = \left[\begin{array}{cc} 0 & I \\ I & 0 \end{array} \right] J^{-1} \left[\begin{array}{cc} 0 & I \\ I & 0 \end{array} \right] = \left[\begin{array}{c|cc} A & L & B_2 \\ \hline -F & 0 & I \\ C_2 & I & D_{22} \end{array} \right].$$

Note that Q is stable if and only if K stabilizes \hat{J}_{22} . But $\hat{J}_{22} = G_{22}$, so Q is stable if and only if K stabilizes G , as desired. We summarize this result as

Theorem 11 *Any admissible controller $K(\Delta)$ can be characterized as a LFT of a Q -stable parameter matrix $Q(\Delta)$, i.e., $K(\Delta) = \mathcal{F}_\ell(J(\Delta), Q(\Delta))$ with $Q(\Delta)$ realized by*

$$Q(\Delta) := \mathcal{F}_\ell(\hat{J}(\Delta), K(\Delta))$$

where

$$\hat{J}(\Delta) = \left[\begin{array}{c|cc} A & L & B_2 \\ \hline -F & 0 & I \\ C_2 & I & D_{22} \end{array} \right]$$

and the realization for $K(\Delta)$ is Q -stabilizable and Q -detectable. Moreover, this characterization is unique for a given pair F and L satisfying the requirements stated in the above theorem.

Remark 11 *Note that the key technique used in the stabilizing controller parametrization for both the disturbance feedforward and the output feedback problem is inversion property of linear fractional transformation (Proposition 2).*

7 Concluding Remarks

We have considered the problems of analysis, stabilization and the parametrization of all stabilizing controllers for LFT systems. All of the manipulations have been based on the definition of stabilities for this kind of systems. The focus has been on Q -stability, most of the results, including the separation theory, also hold in the μ -stability case via simple change of notation. An exception is that the stabilization for FI structure by dynamic feedback is not equivalent to stabilization by constant gain.

The separation property discussed in this paper holds in greater generality than for just the Q and μ stability problems. All that is required for the separation proof is that the notion of stability satisfy two requirements: 1) stability invariance under a sufficiently rich set of similarity transformations, as in Theorem 2, and 2) a certain structural property as given in Theorem 3. It would clearly be possible to develop a more abstract axiomatic stabilization theory using these 2 properties.

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